
I. (16 points) The acceleration $\vec{a}$ of a particle moving in one dimension is depends on time t and is given by

$$\vec{a} = -\left(3.0\,\mathrm{m/s^4}\right) t^2 + \left(18.0\,\mathrm{m/s^3}\right) t$$

If the particle starts at position $x = 2.0\,\mathrm{m}$ at $t = 0$ with an initial velocity of $2.5\,\mathrm{m/s}$, what is its displacement after 2.0 seconds? (Hint: first find the velocity.)

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As $\vec{a} = d\vec{v}/dt$ and $\vec{v} = d\vec{x}/dt$, the velocity can be found as the time integral of the acceleration, and the position can be found as the time integral of the velocity.

As the object is moving in one dimension, signs are sufficient to indicate direction, and vector notation will be dropped. Let $A = -3.0\,\mathrm{m/s^4}$ and $B = 18.0\,\mathrm{m/s^3}$, so the acceleration is

$$a = At^2 + Bt$$

Then the velocity is

$$v = \int a\,dt = \int \left(At^2 + Bt\right) dt = \frac{A}{3}t^3 + \frac{B}{2}t^2 + C$$

where C is an integration constant. When $t = 0$, the velocity is $2.5\,\mathrm{m/s}$, so

$$2.5\,\mathrm{m/s} = \frac{A}{3}0^3 + \frac{B}{2}0^2 + C \qquad \Rightarrow \qquad C = 2.5\,\mathrm{m/s}$$

The position is

$$x = \int \left(\frac{A}{3}t^3 + \frac{B}{2}t^2 + C\right) dt = \frac{A}{3\cdot 4}t^4 + \frac{B}{2\cdot 3}t^3 + Ct + D$$

where D is an integration constant. Displacement is the difference in position from zero to $2.0\,\mathrm{s}$, so

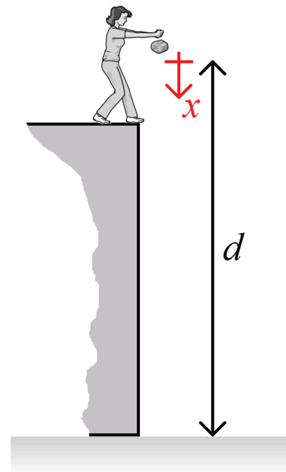
$$\Delta x = x(2.0\,\mathrm{s}) - x(0\,\mathrm{s}) = \left[\frac{A}{12}\left(2.0\,\mathrm{s}\right)^4 + \frac{B}{6}\left(2.0\,\mathrm{s}\right)^3 + C\left(2.0\,\mathrm{s}\right) + D\right] - \left[\frac{A}{12}\left(0\,\mathrm{s}\right)^4 + \frac{B}{6}\left(0\,\mathrm{s}\right)^3 + C\left(0\,\mathrm{s}\right) + D\right]$$

$$= \frac{A}{12}\left(2.0\,\mathrm{s}\right)^4 + \frac{B}{6}\left(2.0\,\mathrm{s}\right)^3 + C\left(2.0\,\mathrm{s}\right)$$

$$= \frac{-3.0\,\mathrm{m/s^4}}{12}\left(16.0\,\mathrm{s^4}\right) + \frac{18.0\,\mathrm{m/s^3}}{6}\left(8.0\,\mathrm{s^3}\right) + 2.5\,\mathrm{m/s}\left(2.0\,\mathrm{s}\right) = -4.0\,\mathrm{m} + 24\,\mathrm{m} + 5.0\,\mathrm{m}$$

$$= +25\,\mathrm{m}$$

II. (16 points) Amber stands on top of a small building, and throws a rock straight downward. It strikes the ground 2.4 s after she releases it, traveling at 25 m/s. At what distance d above the ground did she release it? (*On Earth.*)



After release, the rock moves under the influence of only gravity. This is a free-fall situation, which is a special case of constant acceleration. As the object is moving in one dimension, signs are sufficient to indicate direction, and vector notation will be dropped.

$$x = x_0 + v_0\,\Delta t + \tfrac{1}{2}a\,(\Delta t)^2$$

$$v = v_0 + a\,\Delta t$$

Choose a coordinate system. I've chosen the origin to be at the release point, with positive downward. With this choice,

$$x_0 = 0\,\text{m} \qquad v = +25\,\text{m/s} \qquad a = g = +9.8\,\text{m/s}^2 \qquad \Delta t = +2.4\,\text{s}$$

The initial velocity of the stone, v_0, is unknown, and the final position, x, is d, the answer to the question. Since the initial velocity is unknown, eliminate it from the two constant-acceleration equations.

$$v_0 = v - a\,\Delta t \quad \text{so} \qquad x = x_0 + (v - a\,\Delta t)\,\Delta t + \tfrac{1}{2}a\,(\Delta t)^2$$

Then

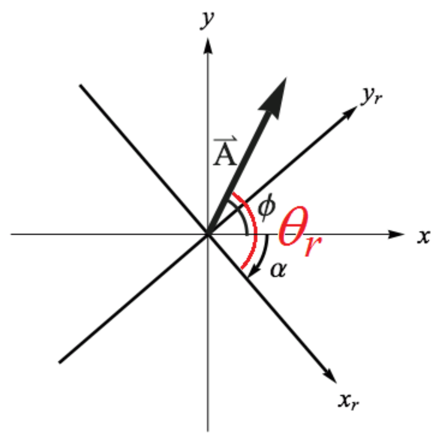
$$\begin{aligned} x &= x_0 + v\,\Delta t - \tfrac{1}{2}a\,(\Delta t)^2 \\ &= (0\,\text{m}) + (25\,\text{m/s})\,(2.4\,\text{s}) - \tfrac{1}{2}\left(9.8\,\text{m/s}^2\right)(2.4\,\text{s})^2 \\ &= 32\,\text{m} \end{aligned}$$

1. (6 points) In the problem above, if Amber had thrown the rock upward with the same initial speed ...

If Amber throws the rock upward, it will reach a maximum height and then descend. As it passes the point at which it was thrown, it will be traveling at the same speed with which it was thrown. From this point on, the motion is exactly the same as the problem above. Some non-zero time, however, was required for the rock to reach this point. So ...

the rock would be traveling at 25 m/s when it reached the ground,
but it would require more than 2.4 s to reach the ground from the time it was thrown.

2. (6 points) The figure shows two coordinate systems, the original coordinate system (x, y), and (x_r, y_r) which has been rotated by α degrees *clockwise* with respect to the original coordinate system. The vector $\vec{A}$ has magnitude and angle $(r, \theta) = (A, \phi)$ in the original coordinate system.



What is the angle θ_r of polar coordinate (r_r, θ_r) of $\vec{A}$ in the rotated coordinate system?

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From the diagram,

$$\theta_r = \alpha + \phi$$

III. (16 points) Write an expression for the Cartesian coordinates (x_r, y_r) of $\vec{A}$ in the rotated coordinate system, in terms of parameters defined in the problem and physical or mathematical constants.

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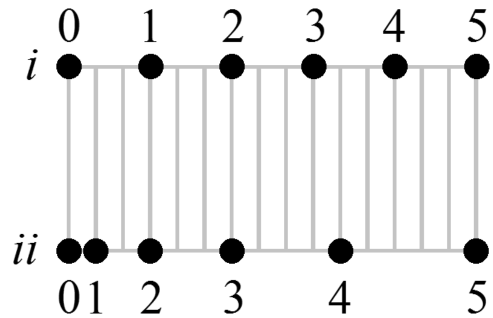
As A, the magnitude of the vector $\vec{A}$, does not depend on the coordinate system, it must be the same in the rotated system. As $\vec{A}$ makes an angle θ_r with the x_r axis of the rotated coordinate system,

$$\vec{A} = (A, \theta_r) = \langle A\cos\theta_r, A\sin\theta_r\rangle$$

3. (8 points) Motion diagrams for two objects, i and ii, are shown. At an instant in what interval, if any, must the objects have the same velocity?



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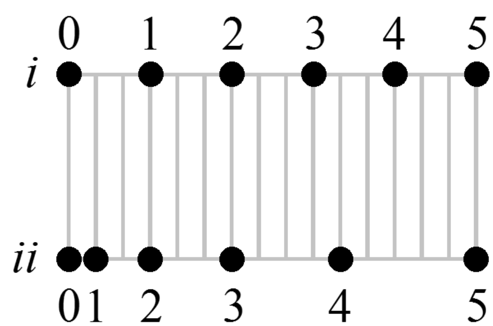
In the intervals 0 to 1, and 1 to 2, the average velocity of object ii is less than that of object i. Even if the accelerations of the objects are wildly non-uniform, there's no reason the velocities *must* be the same in either of these two intervals. The same could be said about the intervals 3 to 4, and 4 to 5, in which the average velocity of object ii is greater than that of object i.

However, in the interval 2 to 3, the objects have the same average velocity. At some point in the interval, each object's instantaneous velocity must be the same as this average velocity. So at some point, the objects must have the same instantaneous velocity

In the interval from 2 to 3.

4. (8 points) Motion diagrams for two objects, i and ii, are shown. At an instant in what interval, if any, must the objects have the same acceleration?



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Object i could have a constant velocity, or an acceleration of zero. Object ii does not have a constant velocity, so there is no necessity for it to have an acceleration of zero at any instant in any interval. So

There is no such instant in any time interval.

5. (8 points) The graph shows the velocity of an object moving in one dimension as a function of time. At what time in the range 0 to 6 s does the object have its greatest displacement from its position at time $t = 0\,\mathrm{s}$?

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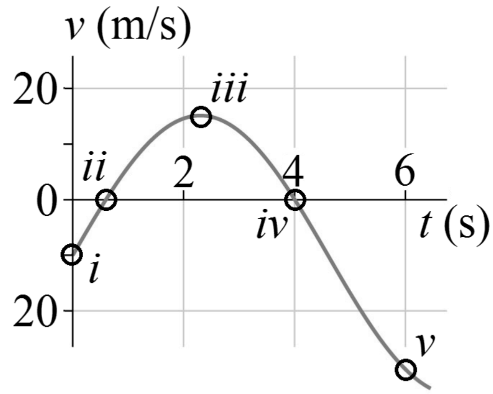

As $x = \int v\,dt$, the displacement is represented by the area under the curve. The greatest displacement will occur at the instant when the greatest area under the curve has been achieved. There is a small amount of negative area between instants *i* and *ii*, then a large amount of positive area between instants *ii* and *iv*, and finally more negative area between instants *iv* and *v*. So, the greatest area under the curve, and thus the greatest displacement, occurs

At $t = 4\,\mathrm{s}$, instant *iv*.

6. (8 points) I drop a stone from a tall tower at time $t = 0$, then one second later I drop another stone. Which velocity *vs.* time diagram describes this situation. ("Up" is the positive direction.)

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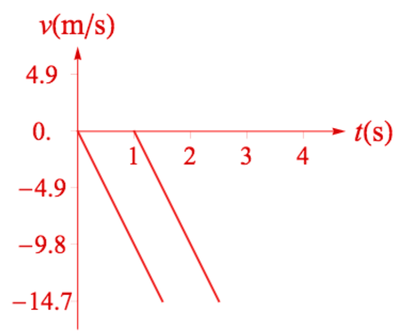

The two stones are both in free fall, so they have the same constant acceleration. Both velocity *vs.* time curves must have the same constant slope. Further, that slope must be (with positive defined upward) $-9.8\,\mathrm{m/s^2}$. Defining positive upward also means that the velocities of the dropped (not thrown) stones must always be positive.

7. (8 points) I am trying to land my plane due north at 5 m/s in a 3 m/s wind 30° north of east. If my target velocity is $\vec{v}_t$, and the wind has velocity $\vec{v}_w$, the velocity I should aim my plane, $\vec{v}_p$, satisfies $\vec{v}_t = \vec{v}_w + \vec{v}_p$. Which diagram of vector addition shows the correct vector along which I should aim my plane.

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When adding two vectors graphically, they must be placed tip-to-tail. The vector representing the sum points from the tail of the first vector to the tip of the second. In this case, $\vec{v}_w$ and $\vec{v}_p$ are the vectors that must be placed tip-to-tail, and $\vec{v}_t$ must lie from the tail of $\vec{v}_w$ to the tip of $\vec{v}_p$.