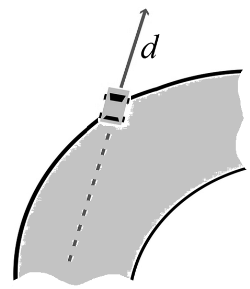
I. (16 points) Corentine is driving her car of mass m around a curve when suddenly, all systems fail! The engine quits, she can't brake, she can't steer, and the car coasts straight off the side of the road, as pictured. Fortunately, the car goes into a plowed field. As the tires sink deeper into the soft dirt, the frictional force magnitude f on the car increases according to

$$f = f_0 \frac{x}{x_0}$$

where x is the distance from where the car left the road, and f_0 and x_0 are positive constants. If the car travels a distance d before stopping in the field, how fast was the car going when it left the road? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants. (*On Earth.*)



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Use the Energy Principle.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Since friction will produce a thermal energy change, choose a system that includes both surfaces, the car and the Earth. With that choice, there is no work done by external forces, and there are no conservative internal forces producing potential energy changes. The Earth's kinetic energy change is negligible. The internal dissipative force varies with position, so an integral must be evaluated to find the thermal energy change.

$$0 = \Delta K + 0 + \Delta E_{\text{th}} = \left(\tfrac{1}{2}mv_{\text{f}}^2 - \tfrac{1}{2}mv_{\text{i}}^2\right) + \int f\,dx$$

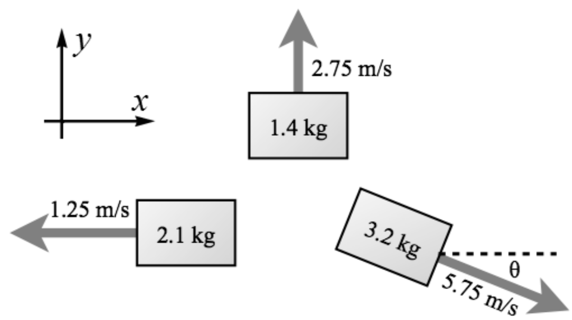
The car travels from $x = 0$ to $x = d$, then comes to a stop with a final speed of zero. Substitute the expression for the friction force.

$$\tfrac{1}{2}mv_{\text{i}}^2 = \int_0^d f_0 \frac{x}{x_0}\,dx = \frac{f_0}{x_0}\int_0^d x\,dx = \frac{f_0}{x_0}\left[\frac{x^2}{2}\right]_0^d = \frac{f_0 d^2}{2x_0}$$

Solve for the initial speed.

$$v_{\text{i}}^2 = \frac{f_0 d^2}{m x_0} \qquad \Rightarrow \qquad v_{\text{i}} = d\sqrt{\frac{f_0}{m x_0}}$$

II. (16 points) An object traveling at initial velocity v_0 explodes into three pieces. Instantly after the explosion the velocities of the pieces are measured. The 2.1 kg piece has velocity $v_1 = -1.25\,\hat{x}\,\mathrm{m/s}$, the 1.4 kg piece has velocity $v_2 = 2.75\,\hat{y}\,\mathrm{m/s}$, and the 3.2 kg piece has speed 5.75 m/s and is traveling at $\theta = 22°$ below the $+x$ axis. What was the velocity of the object immediately before the explosion? (Write your answer in component form using the unit vectors $\hat{x}$ and $\hat{y}$.)



No significant external forces act on the object during the explosion, so momentum is conserved. Since momentum is a vector, when it is conserved, it must be conserved independently on each axis.

$$\Delta\vec{P} = 0 \qquad \vec{P}_\mathrm{i} = \vec{P}_\mathrm{i} \qquad P_\mathrm{ix} = P_\mathrm{fx} \qquad P_\mathrm{iy} = P_\mathrm{fy}$$

Let M represent the total mass. Considering the x direction,

$$Mv_{0x} = m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x}$$

Solve for v_{0x}, not that $v_{2x} = 0$.

$$v_{0x} = \frac{m_1 v_{1x} + m_3 v_{3x}}{M} = \frac{(2.1\,\mathrm{kg})\,(-1.25\,\mathrm{m/s}) + (3.2\,\mathrm{kg})\,(5.75\,\mathrm{m/s})\cos(-22°)}{2.1\,\mathrm{kg} + 1.4\,\mathrm{kg} + 3.2\,\mathrm{kg}} = +2.2\,\mathrm{m/s}$$

Looking next at the y direction,

$$Mv_{0y} = m_1 v_{1y} + m_2 v_{2y} + m_3 v_{3y}$$

Solve for v_{0y}, noting that $v_{1y} = 0$.

$$v_{0y} = \frac{m_2 v_{2y} + m_3 v_{3y}}{M} = \frac{(1.4\,\mathrm{kg})\,(2.75\,\mathrm{m/s}) + (3.2\,\mathrm{kg})\,(5.75\,\mathrm{m/s})\sin(-22°)}{2.1\,\mathrm{kg} + 1.4\,\mathrm{kg} + 3.2\,\mathrm{kg}} = -0.45\,\mathrm{m/s}$$

So

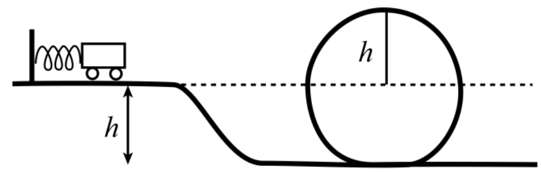
$$\vec{v}_0 = 2.2\,\hat{x} - 0.45\,\hat{y}\,\mathrm{m/s}$$

1. (6 points) After the explosion, the first piece (2.1 kg traveling at $v_1 = -1.25\,\hat{x}\,\mathrm{m/s}$) has a collision with a stationary 625 kg truck (on a frictionless surface). Assuming the collision is completely elastic, what is the velocity of the 2.1 kg object immediately following the collision? (Hint: You don't need to do any calculations.)

As the mass of the truck is much greater than the mass of the first piece, the piece essentially bounces back in an elastic collision, at

$$1.21\,\mathrm{m/s}$$

III. (16 points) Some modern roller coasters launch the riders at the beginning. A roller coaster of mass m starts at a height h above the ground and must go through a loop of radius h such that the top of the loop is twice as high as the launch point. If we model the launch mechanism as an elastic spring with Hooke's Law or Spring Constant k, how much would the launch spring have to be compressed to ensure that the roller coaster has sufficient velocity to get around the circular loop? Express your answers only in terms of m, g, h and k. (*On Earth, neglect friction.*)



Use Newton's Second Law to find the minimum speed that the cart must have at the top of the loop. A Free Body Diagram will have the gravitational force mg downward, and no other force at the minimum speed. Choosing an axis that points downward toward the center of the loop,

$$\sum F_c = mg = ma_c = m\frac{v^2}{r} \qquad \Rightarrow \qquad g = \frac{v^2}{h} \qquad \Rightarrow \qquad v = \sqrt{gh}$$

Now use the Energy Principle.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

I'll choose a system that includes the cart, the Earth, and the spring. With that choice, no external forces do work on the system, and no internal dissipative forces change its thermal energy. The Earth's kinetic energy change is negligible. Both an internal gravitational force and in internal spring force change the system's potential energy.

$$0 = \Delta K + \Delta U_g + \Delta U_s + 0 = \left(\tfrac{1}{2}mv_{\text{f}}^2 - \tfrac{1}{2}mv_{\text{i}}^2\right) + (mgh_{\text{f}} - mgh_{\text{i}}) + \left(\tfrac{1}{2}k\,\Delta s_{\text{f}}^2 - \tfrac{1}{2}k\,\Delta s_{\text{i}}^2\right)$$

The car isn't moving before it is launched ($v_{\text{i}} = 0$). The spring is relaxed after the car is launched (Δs_{f}). I'll choose the initial height to be zero ($h_{\text{i}} = 0$ and $h_{\text{f}} = h$).

$$0 = \left(\tfrac{1}{2}mv_{\text{f}}^2 - 0\right) + (mgh - 0) + \left(0 - \tfrac{1}{2}k\,\Delta s_{\text{i}}^2\right)$$

Substitute the expression for the speed at the top of the loop found above, and solve for the initial compression.

$$\tfrac{1}{2}k\,\Delta s_{\text{i}}^2 = \tfrac{1}{2}m\left(\sqrt{gh}\right)^2 + mgh \qquad \Rightarrow \qquad k\,\Delta s_{\text{i}}^2 = mgh + 2mgh = 3mgh \qquad \Rightarrow \qquad \Delta s_{\text{i}} = \sqrt{\frac{3mgh}{k}}$$

2. (6 points) At what point, if any, in the launch is the power output from the launch spring at its maximum?

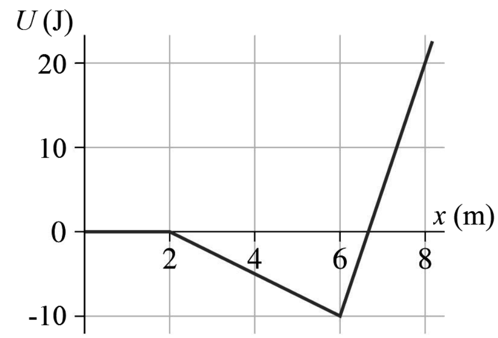
Power is the time rate of energy transformation, which would include the time rate of work being done. (Conservative forces can do work, and thus produce power.)

$$P = \frac{dW}{dt} = \frac{d}{dt}\vec{F}\cdot\vec{s} = \vec{F}\cdot\frac{d\vec{s}}{dt} = \vec{F}\cdot\vec{v}$$

At the beginning of the launch, the speed (and thus the power) is zero. At the end, the spring force (and thus the power) is zero. The power can't be constant if it's zero at the beginning and the end. The only remaining choice offered is that the power is maximum

Sometime in the middle of the launch.

3. (8 points) The graph shows the potential energy, U, of a system as a function of the position, x, of a 2 kg particle within it. This 2 kg particle is released from rest at $x = 8$ m. If the particle can be found later at $x = 4$ m, what speed would it have at that location?



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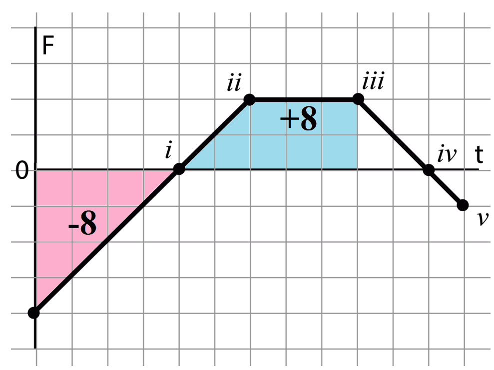
The potential energy of the system is 20 J when the particle is at 8 m, and -5 J when the particle is at 4 m. The particle, therefore, gains 25 J of kinetic energy. Since it started with at rest, it must have 25 J of kinetic energy when it is at 4 m.

$$K = \tfrac{1}{2}mv^2 \qquad \Rightarrow \qquad v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2\,(25\,\mathrm{J})}{2\,\mathrm{kg}}} = 5\,\mathrm{m/s}$$

So

The particle moves at 5 m/s when it is at $x = 4$ m.

4. (8 points) Voyager 2 approaches Saturn and its moon Titan. It is acted upon by a net force from the planet and its moon plotted in the figure. If Voyager 2 has a momentum of 10^7 kg·m/s at $t = 0$ s, at what later point in time does Voyager 2 have the same momentum?



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Impulse, which changes momentum, is

$$\vec{J} = \int \vec{F}\,dt$$

which is represented as the area under a graph of force as a function of time. For Voyager 2 to have the same momentum at a later time, the impulse, and thus the area under the force-time graph, must be zero. Counting squares, we see -8 "units" of impulse between zero and i. +8 "units" of impulse, bringing the total to zero, occur between time i and time

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5. (8 points) A ball of mass m is tossed vertically in the air with velocity v_0. It eventually reaches a height h_0. If the zero of potential energy for the Earth-ball system is chosen at the point the ball is released, what are the ball's height h and velocity v when the total energy of the system is comprised precisely of half potential energy and half kinetic energy? (*On Earth. Air resistance is negligible.*)

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When the system's energy is half kinetic and half potential, each must be half the total energy.

With the given choice of zero point for potential energy, when the ball is tossed, the energy of the system is entirely kinetic. When the kinetic energy is half as much,

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}\left[\tfrac{1}{2}mv_0^2\right] \qquad \Rightarrow \qquad v = v_0/\sqrt{2}$$

When the ball is at height h_0 the system's energy is entirely potential. When the kinetic energy is half as much,

$$mgh = \tfrac{1}{2}\left[mgh_0\right] \qquad \Rightarrow \qquad h = h_0/2$$

So

$$h = h_0/2 \text{ and } v = v_0/\sqrt{2}$$

6. (8 points) The Navy uses aircraft catapults to launch fighter jets from aircraft carriers. The aircraft catapult can be modeled by a giant spring. Two jets of mass m_1 and $m_2 = 3m_1$ are launched using the same spring, displaced by the same amount $\Delta s = d$. Compare the work done by the spring on the second plane W_2 to that done on the first plane W_1.

. . . .

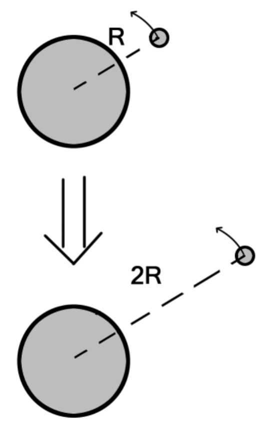
As

$$W = \int \vec{F} \cdot d\vec{s}$$

and the same spring is displaced by the same amount,

$$W_2 = W_1$$

7. (8 points) A satellite goes around the Earth in a circular orbit with radius R. Its kinetic energy is E_k. A rocket engine is turned on, increasing the energy of the satellite. After this, the satellite orbits with a new radius $2R$. What is the new kinetic energy of the satellite in terms of E_k?



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From Newton's Second Law, we know

$$\sum F_c = F_g = ma_c \qquad \Rightarrow \qquad \frac{GMm}{r^2} = m\frac{v^2}{r} \qquad \Rightarrow \qquad \tfrac{1}{2}mv^2 = \frac{GMm}{2r}$$

If the zero of potential energy is chosen at infinite separation, then

$$U = \frac{-GMm}{r} \qquad \Rightarrow \qquad K = -U/2$$

So when the radius of the orbit is doubled, the potential energy is halved, *and* the kinetic energy is halved. The new kinetic energy will be

$$E_k/2$$